ON THE STABILITY OF MOTION IN CASE OF TWO SMALL POSITIVE ROOTS

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We consider the stability of a dynamic system, the motion of which is described by a system of differential equations of the type

$$x^{\cdot} = \mu x + X(x, y), \qquad y^{\cdot} = \mu a y + Y(x, y) \tag{0.1}$$

Here μ is a small positive number, $a \geq 0$ and $X(x, y)$ and $Y(x, y)$ are holomorphic functions in the vicinity of the unperturbed motion $x = y = 0$, the expansion of which contains no terms of higher than second order and which can be represented as a sum

$$X(x, y) = X^{(m)}(x, y) + X^{(m+1)}(x, y) + \ldots, \; Y(x, y) = Y^{(m)}(x, y) + Y^{(m+1)}(x, y) + \cdots$$

The characteristic equation of the system (0. 1) has two small positive distinct roots, $\varkappa_1 = \mu$ and $\varkappa_2 = \mu a$.

1. Following [1], we shall call the cases when the characteristic equation has the right-hand roots of low absolute value, the cases near to critical. Presence of positive, although small roots, is a necessary and sufficient condition for the instability of perturbed motion in the Liapunov sence [2], independent of nonlinear terms. These conclusions were reached, when the restraints imposed on the magnitudes of initial deviations were very strong.

If the system admits the deviations exceeding in magnitude those allowed in the above mentioned theorem, then the problem of stability becomes open.

In the problems of this type we shall, when investigating the stability, make use of the definition of stability formulated by Kamenkov [1].

"If, in the space $x_1, \ldots, x_n$ a closed region G can be found possessing the property that the perturbations $x_1, \ldots, x_n$ assumed to be functions of time and satisfying the equations of perturbed motion do not emerge outside this region for any $t \geq t_0$ if only their initial values $x_{10}, \ldots, x_{n0}$ were contained within this region or on its boundary, then the unperturbed motion will be stable ; otherwise it will be unstable.

It may happen, and this is a general case, that inside the region G there exists another closed region G_1 and relative to it, the motion may be unstable. The cases when several such regions may be present and enclosed within each other, are not excluded".

Let us consider Equations (0. 1) in polar coordinates $x = r\cos\theta$ and $y = r\sin\theta$

$$r' = \mu r\,[\cos^2\theta + a\sin^2\theta] + r^m R_0(\theta) + r^{m+1} R_1(\theta) + \ldots$$
$$r\theta' = \mu r\,(a-1)\sin\theta\cos\theta + r^m F_0(\theta) + r^{m+1} F_1(\theta) + \ldots \qquad (1.1)$$

Here

$$R_k(\theta) = X^{(m+k)}(\cos\theta,\ \sin\theta)\cos\theta + Y^{(m+k)}(\cos\theta,\ \sin\theta)\sin\theta$$
$$F_k(\theta) = Y^{(m+k)}(\cos\theta,\ \sin\theta)\cos\theta - X^{(m+k)}(\cos\theta,\ \sin\theta)\sin\theta$$

are complete rational functions of $\cos\theta$ and $\sin\theta$.

Setting $\mu = 0$ results in the problem of two zero roots with two groups of solutions, which has a thorough treatment in the papers by Kamenkov [3 and 4]. In case of stable motions, the problem can be investigated by two distinct methods depending on whether the function $F_0(\theta)$ is sign definite, or whether it can become zero on the interval $[0,\ 2\pi]$. The latter method is, in our case, preferable.

Let us take the Liapunov function in the form

$$V = r\exp\int_0^\theta \Psi(\theta)\,d\theta \qquad \left(\int_0^{2\pi} \Psi(\theta)\,d\theta = 0\right) \qquad (1.2)$$

where $\Psi(\theta)$ is a continuous periodic function to be defined.

The derivative of V is, in accordance with (1.1),

$$V' = \exp\int_0^\theta \psi(\theta)\,d\theta\,\{\mu r\,[\cos^2\theta + a\sin^2\theta + \psi(\theta)(a-1)\sin\theta\cos\theta] +$$
$$+ r^m\,[R_0(\theta) + \psi(\theta)F_0(\theta)] + r^{m+1}\,[R_1(\theta) + \psi(\theta)F_1(\theta)] + \ldots\} \qquad (1.3)$$

2. Let us consider two cases, in which the problem of stability is solved with the help of mth order forms

$$R_0(\theta) < 0 \quad \text{for } F_0(\theta) = 0, \qquad 0 \leqslant \theta \leqslant 2\pi \qquad (2.1)$$

$$gF_0(\theta) < 0, \qquad g = \int_0^{2\pi} \frac{R_0(\theta)}{F_0(\theta)}\,d\theta \neq 0 \quad \text{for } F_0(\theta) \neq 0, \qquad 0 \leqslant \theta \leqslant 2\pi \qquad (2.2)$$

In both cases the motion is asymptotically stable by [3], and this paper also shows the possibility of choosing $\Psi(\theta)$ while satisfying (2.1) and gives the method for its construction. It also proves that a function $\Psi(\theta)$ satisfying

$$R_0(\theta) + \Psi(\theta)F_0(\theta) \leqslant -M \qquad (2.3)$$

where $M > 0$ is a constant, can always be found. We should note that in the majority of practical problems m is found to be fairly small (between 3 and 5). This allows us to select the function $\Psi(\theta)$ using simple graphical constructions. One should however make the attempt to obtain M as large as possible, otherwise large errors may arise during the determination of the boundary of the region G_*

If the functions $R_0(\theta)$ and $F_0(\theta)$ satisfy the conditions (2.1), then in addition to the method given above we can use another, strictly analytical method to obtain $\Psi(\theta)$

$$\psi(\theta) = g - \frac{R_0(\theta)}{F_0(\theta)}$$

Condition (2.3) then becomes $gF_0(\theta) \leq -M$. Choosing $\Psi(\theta)$ in one way, or the other, we fully define the function

$$\cos^2\theta + a\sin^2\theta + \Psi(\theta)(a-1)\cos\theta\sin\theta$$

Let us choose its largest value

$$\delta = \sup\,[\cos^2\theta + a\sin^2\theta + \Psi(\theta)(a-1)\cos\theta\sin\theta] \qquad (0 \leqslant \theta \leqslant 2\pi)$$

We shall use M and δ in majorizing the corresponding terms in (1. 3)

$$V' < \exp \int_0^\theta \psi(\theta)\, d\theta\, [\mu r\delta - r^m M + r^{m+1}(\ldots) + \ldots]$$

from which we can obtain the boundary of G_* interior to G

$$r_* = \left(\frac{\mu\delta}{M}\right)^{\frac{1}{m-1}} \tag{2.4}$$

It can be asserted that when $r > r_*$, then $V' < 0$ is independent of terms of the order higher than m, provided that r and r_* are sufficiently small. Consequently, on one hand there exists a closed cycle $V = C$ intersected by the integral curves of (0. 1) directed inwards, on the other hand the point $x = y = 0$ is an unstable node when $\mu \neq 0$. By the Bendickson theorem, there exists between the coordinate origin and the closed cycle $V = C$, a critical cycle which has a corresponding periodic motion. Relation (2. 4) gives a value which is on the high side and is caused by the majorization, shows that the region G_* appears only when small positive roots are present and shows that the size of this region is directly related to the magnitude of μ. At the same time, the magnitude of r_* can be varied by varying M, i. e. by varying the coefficients of nonlinear terms in the initial system (0. 1).

In this manner, the motion unstable in the Liapunov sense, is found to be stable, in some region, in the sense of the definition quoted above, provided the motion given by the critical system ($\mu = 0$) is asymptotically stable in the Liapunov sense. If, on the other hand, the motion described by the critical system is unstable or not asymptotically stable, then the appearance of small positive roots will certainly make it unstable.

Two particular cases must be singled out. In the first case we have $a = 1$. The initial system will have a small double positive root with two groups of solutions. Magnitude r_* will be of the form (2. 4) with $\delta = 1$. Second case corresponds to $a = 0$, which is arrived at whenever the system of differential equations under investigation has a characteristic equation with two complex, conjugate roots with small positive real parts, and another root equal to zero.

3. Above, we have investigated the cases when the characteristic equation of (0. 1) contained only small positive real roots.

Next, we shall consider a system of equations of a more general type

$$x^{\cdot} = \mu\,(a_1 x + a_2 y) + X\,(x, y), \qquad y^{\cdot} = \mu\,(b_1 x + b_2 y) + Y\,(x, y) \tag{3.1}$$

With the conditions $a_1 + b_2 > 0$ and $a_1 b_2 - a_2 b_1 > 0$, we shall single out two cases.

1) $(a_1 - b_2)^2 + 4a_2 b_1 \geqslant 0$ when the characteristic equation has small real positive roots. This case can be reduced to one of the cases considered previously.

2) $(a_1 - b_2)^2 + 4a_2 b_1 < 0$ when the characteristic equation has a pair of conjugate complex roots of small modulus and possessing positive real parts.

For the method used to solve the present problem, it is immaterial which of the above cases takes place. Hence, applying the above to (3. 1) we can conclude, that, in contrast with the previous calculations, only δ will change, and it must, after the selection of $\Psi(\theta)$, be obtained from

$$\delta = \sup\,\{a_1 \cos^2\theta + b_2 \sin^2\theta + (a_2 + b_1)\sin\theta\cos\theta + \Psi(\theta)\,[b_1\cos^2\theta - a_2\sin^2\theta + (b_2 - a_1)\cos\theta\sin\theta]\} \qquad (0 \leqslant \theta \leqslant 2\pi) \tag{3.2}$$

The rest of the arguments remain fully in force.

We should note that if the characteristic equation of the investigated system, apart from small positive roots, also has negative roots and roots with negative real parts, we can employ Kamenkov's [3] idea of separating the critical and noncritical variables, which in turn allows us to investigate either (0.1) or (3.1).

4. As an example, we shall consider the problem of stabilization of an object inherently unstable in two coordinates. Stabilization will be effected by a nonlinear control with odd characteristic and possessing a zone of insensitivity. We shall use the following differential equations of perturbed motion of the system

$$\begin{aligned} x_1^{\cdot} &= \mu x_1 - \beta x_2 + n_1 z + X_1(x_1, x_2) \\ x_2^{\cdot} &= \mu x_2 + \beta x_1 + n_2 z + X_2(x_1, x_2) \qquad (0 < \mu \ll 1) \\ z^{\cdot} &= f(\sigma) = S_3\sigma^3 + S_5\sigma^5 + \ldots, \qquad \sigma = k_1x_1 + k_2x_2 + r_0 z \end{aligned} \tag{4.1}$$

where n_1 and n_2 are the parameters of the object giving the measure of the influence of the control on it; S_3, S_5, ... are the parameters of the nonlinear characteristic of the control; σ is the controlling impulse signal and $X_1(x_1, x_2)$, $Y_1(x_1, x_2)$ are holomorphic functions near the coordinate origin, the expansions of which start with second order terms.

Assuming that the parameters of the object are specified, we shall choose the parameters of the control so as to obtain the solution of the stability problem in terms of order not higher than the third.

Changing to new variables

$$z_1 = x_1 + a_1z + a_2z^2 + a_3z^3, \qquad z_2 = x_2 + b_1z + b_2z^2 + b_3z^3, \qquad z = z \tag{4.2}$$

we obtain (4.1) in the form

$$\begin{aligned} z_1^{\cdot} &= \mu z_1 - \beta z_2 + Z_1^{(2)}(z_1, z_2, z) + Z_1^{(3)}(z_1, z_2, z) + \ldots \\ z_2^{\cdot} &= \mu z_2 + \beta z_1 + Z_2^{(2)}(z_1, z_2, z) + Z_2^{(3)}(z_1, z_2, z) + \ldots \\ z^{\cdot} &= Z^{(3)}(z_1, z_2, z) + Z^{(4)}(z_1, z_2, z) + \ldots \end{aligned} \tag{4.3}$$

Here $Z_1^{(2)}, Z_1^{(3)}, Z_2^{(2)}$ and $Z_2^{(3)}$ do not contain terms independent of z_1 and z_2. In cylindrical coordinates $z_1 = r_1\cos\theta$, $z_2 = r_1\sin\theta$ and $z = z$, (4.3) becomes

$$\begin{aligned} r_1^{\cdot} &= \mu r_1 + r_1[Q^{(20)}(\theta)r_1 + Q^{(11)}(\theta)z] + \\ &+ r_1[Q^{(30)}(\theta)r_1^2 + Q^{(21)}(\theta)r_1z + Q^{(12)}(\theta)z^2] + R(r_1, z, \theta) \\ r_1\theta^{\cdot} &= \beta r_1 + r_1[F^{(20)}(\theta)r_1 + F^{(11)}(\theta)z] + \\ &+ r_1[F^{(30)}(\theta)r_1^2 + F^{(21)}(\theta)r_1z + F^{(12)}(\theta)z^2] + \Theta(r_1, z, \theta) \\ z^{\cdot} &= r_1[P^{(30)}(\theta)r_1^2 + P^{(21)}(\theta)r_1z + P^{(12)}(\theta)z^2] + q^{(03)}z^3 + Z(r_1, z, \theta) \end{aligned}$$

where $Q^{(k,s)}(\theta)$, $F^{(k,s)}(\theta)$ and $P^{(k,s)}(\theta)$ are determinate periodic functions of θ. Using the well known interchange with periodic coefficients [3] we can transform these equations into another form, in which the terms of up to and including the third order will have constant coefficients, i.e.

$$\begin{aligned} \rho^{\cdot} &= \mu\rho\,[1 - \sum_{k_1+k_2=1} G^{(k_1k_2)}(\theta)\,\rho^{k_1}\xi^{k_2} + \ldots] + g^{(11)}\rho\xi + g^{(30)}\rho^3 + g^{(12)}\rho\xi^2 + P(\rho, \xi, \theta) \\ \xi^{\cdot} &= \mu\rho\,[-(k_2+1)\sum_{k_1+k_2=2} H^{(k_1k_2)}(\theta)\,\rho^{k_1}\xi^{k_2} + \ldots] + q^{(03)}\xi^3 + q^{(21)}\xi\rho^2 + \Xi(\rho, \xi, \theta) \end{aligned} \tag{4.4}$$

Here $G^{(k_1k_2)}(\theta)$ and $H^{(k_1k_2)}(\theta)$ are known periodic functions of θ, while $P(\rho, \xi, \theta)$ and $\Xi(\rho, \xi, \theta)$ are holomorphic functions near $\rho = \xi = 0$ with the coefficients periodic in θ, and whose expansion does not contain terms of the order lower than the fourth in ρ and ξ.

We shall use (4.4) which is formally analogous to (0.1) when $a = 0$, as a starting point. The problem is separated into two, basically different cases, depending on whether the coefficient $g^{(1,1)}$ is, or is not equal to zero. Let us consider the case of $g^{(1,1)} = 0$. This is equivalent to the condition

$$(A\mu + B\beta)\, n_1 + (B\mu + A\beta)\, n_2 = 0 \tag{4.5}$$

where A and B are known quantities determined from the parameters of the object. If the parameters n_1 and n_2 can be chosen so as to satisfy (4.5), then the problem on stability will be fully solvable in third order terms.

Passing to new variables $\rho = r \sin\varphi$ and $\xi = r\cos\varphi$ in (4.4) and selecting V in the form of (1.2), we can express (1.3) as

$$V' = \exp\int_0^\varphi \psi(\varphi)\, d\varphi\ \{[\mu r(\sin^2\varphi + \psi(\varphi)\sin\varphi\cos\varphi) + \mu r^2(\ldots)] + r^3[R_0(\varphi) + \psi(\varphi) F_0(\varphi)] + r^4[\ldots]\} \tag{4.6}$$

where

$$F_0(\varphi) = [(g^{(30)} - q^{(21)})\sin^2\varphi + (g^{(12)} - q^{(03)})\cos^2\varphi]\sin\varphi\cos\varphi$$

$$R_0(\varphi) = g^{(30)}\sin^4\varphi + (g^{(12)} + q^{(21)})\sin^2\varphi\cos^2\varphi + q^{(03)}\cos^4\varphi$$

Function $F_0(\varphi)$ is not sign definite and may become zero either on two, or on four rays, depending on the relationships existing between the coefficients.

1°. Function $F_0(\varphi) = 0$ on the rays $\varphi_1 = k\pi$ and $\varphi_2 = \frac{1}{2}\pi + k\pi$ ($k = 0, 1, 2, \ldots$). This is possible, if the relationship

$$0 < \frac{g^{(12)} - q^{(03)}}{g^{(30)} - q^{(21)}} = -N \tag{4.7}$$

exists between the coefficients.

Condition (2.1) requires the fulfillment of yet another two inequalities

$$g^{(30)} < 0, \qquad q^{(03)} < 0 \tag{4.8}$$

In this case the choice of parameters of the control must be based on the simultaneous fulfilment of conditions (4.5), (4.7) and (4.8).

2°. Function $F_0(\varphi) = 0$ on the rays φ_1, φ_2 and $\varphi_{3,4} = \tan^{-1} \pm (N)^{\frac{1}{2}}$. Here we must have $N > 0$. Condition (1.2) demands that (4.8) and

$$R_0(N) = g^{(30)}N^2 + (g^{(12)} + q^{(21)})\, N + q^{(03)} < 0 \tag{4.9}$$

be fulfilled.

In this case parameters of the regulator must be chosen from the conditions (4.5), (4.8), (4.9) and $N > 0$. Choice of the $\Psi(\varphi)$ is governed by the properties of $R_0(\varphi)$. If, apart from the conditions already fulfilled, the relation

$$(g^{(12)} + q^{(21)})^2 < 4g^{(30)}q^{(03)} \tag{4.10}$$

can also be satisfied, then $R_0(\varphi)$ will be found to be negative definite. In this case we can put $\Psi(\varphi) \equiv 0$ and use

$$M = \sup R_0(\varphi), \qquad 0 \leqslant \varphi \leqslant 2\pi, \qquad \delta = 1$$

in the determination of r_*.

Such a choice of $\Psi(\varphi)$ can, however, be recommended only when $R_0(\varphi)$ exhibits a narrow range of variation while remaining negative everywhere. Otherwise, an appreciable error may arise in the determination of r_*.

When $R_0(\varphi)$ undergoes a wide range of variation and when the condition (4.10) can not be satisfied, the most suitable form of $\Psi(\varphi)$ is

$$\Psi(\varphi) = a\cos^3\varphi\sin\varphi + b\cos\varphi\sin^3\varphi$$

where a and b are real numbers, one of which may be zero, and which are given by a well defined form of $F_0(\varphi)$ and $R_0(\varphi)$. Such a choice of $\Psi(\varphi)$ will always enable us to select M from the following relationships: (1) $M = \sup[g^{(30)}, q^{(03)}]$ - in the case of two rays and (2) $M = \sup[g^{(30)}, q^{(03)}, R_0(N)]$ - in the case of four rays.

Number δ can be found from

$$\delta = \sup[\sin^2\varphi + a\sin^2\varphi\cos^4\varphi + b\cos^2\varphi\sin^4\varphi] \quad (0 \leqslant \varphi \leqslant 2\pi)$$

The selected numbers enable us to find r_* and this completes the solution of our problem in case of $g^{(1,1)} = 0$.

When the condition (4.5) cannot be satisfied by a suitable choice of parameters n_1 and n_2, the problem becomes much more complex and the terms of the order higher than the third must be brought into use in order to obtain its solution.

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